

RLS-based Identification of fractional order H_{n_1, n_2} system using the Singularity Function approximation

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Abstract: This paper presents a study of fractional order systems modeling and identification by recursive least squares (RLS) with forgetting factor estimation technique. The fractional order integrators are implemented using the Singularity Function approximation method. Parametric Identification of fractional order differential equations (FDE) is investigated when estimating system parameters by a linear model with respect to parameters, as well as non-integer orders from temporal data (H_{n_1, n_2})-type model. A numerical simulation example illustrates the effectiveness of the proposed identification approach to ensure the convergence of the plant and model outputs even if a bias is persistent in parameters' values.

Keywords: System Identification, Recursive Estimation, Recursive Least squares Method, Fractional order Integrator, Singularity Function approximation.

1. INTRODUCTION

The Fractional Calculus (FC) is a generalization of classical calculus concerned with operations of integration and differentiation of non-integer (fractional) order. The theory of fractional calculus developed rapidly, mostly as a foundation for a number of applied disciplines, including fractional differential equations (FDE) and fractional dynamics. The applications of FC are very wide nowadays. Recently, many authors made a great effort to apply this knowledge in practice (physics and engineering science problems). However, some physical problems are still under debate and controversy as well as modeling and identification of fractional system [1].

In contrast to the integer order differential equations which are well studied, analytical solution of (FDEs) is rather complicated, and for higher order equations, almost impossible. The difficulty arises especially when the system order increases.

In this work, we have used the approximation method for fractional integration proposed by Charef [1,2] for modeling and simulation of fractional systems defined by FDE equation. The identification of fractional order systems remains a real challenge in regard to the augmented complexity introduced by the unknown

fractional orders, comparatively to classical integer order systems [3]. Parametric identification of fractional order linear and non linear systems has been widely investigated in literature [4-6]. But, other techniques were also applied for this aim like the modulating functions method [7]. Many research authors have addressed the problem of identification for processes with fractional order models mainly for control purposes [8-10]. In [11], Du et al. propose an evolutionary algorithm called composite differential evolution (CoDE) for the identification of fractional-order systems with unknown initial values and structure. Whereas, in [12], the Particle Swarm Optimization algorithm is used for this aim. In [13], Ghanbari and Haeri proposed the identification of fractional-order systems through orthogonal rational functions using a fractional Legendre basis and in [14], Li et al. propose a parameter identification of fractional order linear system based on Haar wavelet operational matrix, whereas Abuaiasha and Kertzschler [15] used fractional-order modes for parameter identification of electrical coils. More recently Lu et al. [16] proposed a method for parameter identification of fractional order systems with nonzero initial conditions based on block pulse functions.

The work of this paper provides a way to identify parameters of the $(H_{n1,n2})$ model [8,9]. The recursive least squares (RLS) with forgetting factor using to estimate parameters of linear model.

This paper is organized as follows: Section 2 presents some mathematical basics of fractional calculus and the considered approximation approach for fractional order functions. Section 3 presents a detailed and description of the fractional integration operator, the singularity function method of Charef. Section 4 presents fractional differentials equation FDE.

Section 5 presents the recursive least squares (RLS) with forgetting factor algorithm used to estimation the parameters of the model $(H_{n1,n2})$. The identification fractional system and FDE after model linearization and numerical simulation are presented in section 6 and 7. Finally, conclusions are given in Section 8.

2. FRACTIONAL INTEGRATION

There exist many formulations for the fractional order integration definition; the most popular are those of Grünwald-Letnikov (GL), Riemann-Liouville (RL) and Caputo [10,17].

Riemann-Liouville (RL) fractional order integral is expressed as:

$$I_n f(t) = \frac{1}{\Gamma(n)} \int_0^t (t - \tau)^{n-1} f(\tau) d\tau \quad (1)$$

when $\Gamma(n)$ is the gamma function

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx \quad (2)$$

If $I_n(f(t))$ is convolution of function with the impulse response:

$$h_n(t) = \frac{t^{n-1}}{\Gamma(n)} \quad (3)$$

On the fractional operator whose Laplace transform is

$$I_n(s) = L\{h_n(t)\} = \frac{1}{s^n} \quad (4)$$

Notice that in the integer order case ($n=1$), the integral is characterized by $h_1(t)=H(t)$ (unit step function or Heaviside function):

$$I_1(s) = L\{h_1(t)\} = \frac{1}{s} \quad (5)$$

Fractional differentiation is the dual operation of fractional integration.

$$x(t) = I_n(v(t)) \text{ or } X(s) = \frac{1}{s^n} V(s) \quad (6)$$

Reciprocally, $v(t)$ is the n^{th} order fractional derivative of $x(t)$ defined as :

$$v(t) = D_n(x(t)) \text{ or } V(s) = s^n X(s) \quad (7)$$

Where $D_n(s) = s^n$ represents the Laplace transform of the fractional differentiation operator (for initial conditions equal to zero).

This fractional derivative definition is based on the operator $I_n(s)$, without analytical formulation of $D_n(x(t))$: it is the implicit definition of the fractional derivative.

3. APPROXIMATION METHOD OF THE FRACTIONAL INTEGRAL OPERATOR

This approximation method called singularity function algorithm proposed by Charef allowing us to convert the fractional transfer function into rational one for a prescribed error and a chosen frequency bandwidth [18,19].

A. Fractional order integration

The fractional order integrator is represented by the following irrational transfer function.

$$H(s) = \frac{1}{s^\beta}, 0 < \beta < 1 \quad (8)$$

The singularities (poles, zeros), are computed using following algorithms:

$$P_i = (ab)^i P_0 \quad i = 0, 1, \dots, N \quad (9)$$

$$Z_i = (ab)^i a P_0 \quad i = 0, 1, \dots, N - 1 \quad (10)$$

The approximation analog transfer function is calculated by:

$$H(s) = \frac{K_I}{(1 + \frac{s}{\omega_c})^\beta} = K_I \frac{\prod_{i=0}^{N-1} (1 + \frac{s}{Z_i})}{\prod_{i=0}^N (1 + \frac{s}{P_i})} \quad (11)$$

Where a, b, K_I, P_0, Z_0 are given by:

$$a = 10^{\frac{\epsilon}{10(1-\beta)}} \quad (12)$$

$$b = 10^{\frac{\epsilon}{10\beta}} \quad (13)$$

$$P_0 = \omega_c \sqrt{b} \quad (14)$$

$$Z_0 = a P_0 \quad (15)$$

$$K_I = \frac{1}{\omega_c^\beta} \quad (16)$$

$$\omega_c = \omega_b \sqrt{10^{\frac{\epsilon}{10\beta}} - 1} \quad (17)$$

ϵ is the tolerated error in dB.

The approximation order N is calculated by fixing the working frequency bandwidth $\omega \in [\omega_b, \omega_h]$ such that $\omega \gg \omega_c$

$$N = \text{Integer} \left[\frac{\log(\frac{\omega_{max}}{P_0})}{\log(ab)} + 1 \right] + 1 \quad (18)$$

4. FRACTIONAL DIFFERENTIAL EQUATION MODEL $H_{N1,N2}$

Consider two derivative FDE ($H_{n1,n2}$ model) [3]:

$$a_0 y(t) + a_1 \frac{d^{m_1} y(t)}{dt^{m_1}} + \frac{d^{m_2} y(t)}{dt^{m_2}} = b_0 u(t) \quad (19)$$

m_1 and m_2 are fractional orders verifying $m_1 < m_2$:

$$\text{where } \begin{cases} m_1 = n_1 \\ m_2 = n_1 + n_2 \end{cases} \quad (20)$$

and: $0 < n_1 < 1, 0 < n_2 < 1$

The simulation of the two derivative FDE is based on the fractional integration operator $\tilde{I}_{n_1}(s)$ and $\tilde{I}_{n_2}(s)$. n_1 and n_2 are the respective orders of two integrators required for simulation.

The state space representation corresponding to the FDE defined by (Eq.19) is given by relation (21):

$$\begin{cases} \frac{d^{n_1} x_1(t)}{dt^{n_1}} = x_2(t) \\ \frac{d^{n_2} x_2(t)}{dt^{n_2}} = -a_0 x_1(t) - a_1 x_2(t) + u(t) \\ y(t) = b_0 u(t) \end{cases} \quad (21)$$

Which correspond to the closed loop simulation system of figure 1.

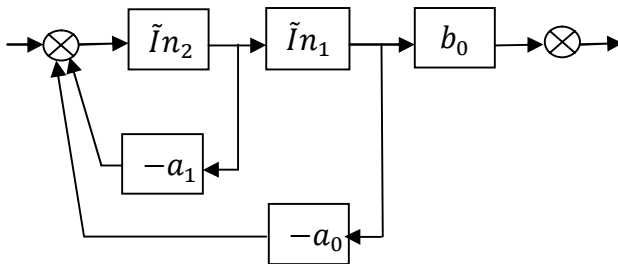


Fig. 1 Block Representation schema of non-integer system

5. RECURSIVE LEAST SQUARES ALGORITHM (RLS)

The recursive least squares method (RLS) [6] is a popular method of for modeling in industrial application, it can be applied to identification using linear model with respect to parameters (Lp) model given by equation (22),

$$\hat{y}(t) = \varphi(t)^T \hat{\theta}(t) \quad (22)$$

where $\varphi(t)$, $\hat{\theta}(t)$ and $\hat{y}(t)$ are regression vector, estimated parameters and estimated measure respectively. By minimizing the

quadratic criterion function J, estimation of, can be obtained

$$J(\hat{\theta}, t) = \sum_{i=1}^t [y(i) - \varphi(i)^T \hat{\theta}(i)]^2 \quad (23)$$

Solving for the minimizing parameters, we get the closed form solution as follows:

$$\hat{\theta}(t) = [\sum_{i=1}^t \varphi(i) \varphi(i)^T]^{-1} \sum_{i=1}^t \varphi(i) y(i) \quad (24)$$

where $y(i)$ represents the measure alternately. The recursive least squares (RLS) handles with algorithm for recursively estimating parameters in linear regression models.

Most of the time, we are interested in real-time parameter estimation. Therefore, it is computationally more efficient if we update the estimates in equation (24) recursively as new data becomes available online. The recursive form is given by:

$$\hat{\theta}(t) = \hat{\theta}(t-1) + p(t) \varphi(t) [y(t) - \varphi^T(t) \hat{\theta}(t-1)] \quad (25)$$

where

$$p(t) = [\sum_{i=1}^t \varphi(i) \varphi(i)^T]^{-1} \quad (26)$$

Note that the term $y(t) - \varphi^T(t) \hat{\theta}(t-1)$ is the prediction error where $\varphi^T(t) \hat{\theta}(t-1)$ is the prediction of $y(t)$ using model variable at time $t-1$.

A. RLS with forgetting factor

In the least square method, forgetting can be viewed as giving less weight to older data and more weight to recent data. The loss-function is then defined as follows:

$$J(\hat{\theta}, t) = \sum_{i=1}^t \lambda^{t-i} [y(i) - \varphi(i)^T \hat{\theta}]^2 \quad (27)$$

Where λ is called forgetting factor $0 < \lambda \leq 1$. The RLS algorithm with forgetting factor we need to implement are:

$$\hat{\theta}(t) = \hat{\theta}(t-1) + K(t) [y(t) - \varphi^T(t) \hat{\theta}(t-1)] \quad (28)$$

$$K(t) = P(t) \varphi(t) = \frac{p(t-1) \varphi(t)}{[\lambda + \varphi^T(t) P(t-1) \varphi(t)]} \quad (29)$$

$$P(t) = \frac{(I - K(t) \varphi^T(t)) p(t-1)}{\lambda} \quad (30)$$

When $\lambda = 1$ this is simply the RLS algorithm.

In order to perform the recursion, we need an initial value for $P(0)$ and initial estimate for $\varphi(0)$, we assume $P(0) = I$ and $\varphi(0) = 0$

Using algorithm with a forgetting factor we may speak about real time identification [20,21].

6. IDENTIFICATION OF FRACTIONAL ORDER SYSTEM

In this section the goal is to find a linear model with respect to parameters, based on repeated integration. For this purpose, the fractional system of the $H_{n1,n2}$ model can be expressed by [22-23]:

$$D_{m2}(y) + a_1 D_{m1}(y) + a_0 y = b_0 u \quad (31)$$

We can write:

$$D_{n1+n2}(y) + a_1 D_{n1}(y) + a_0 y = b_0 u \quad (32)$$

Then by integration of the differential equation with $1/s^{(n1+n2)}$, one obtains:

$$I_{n1+n2}(D_{n1+n2}(y)) + a_1 I_{n1+n2}(D_{n1}(y)) + a_0 I_{n1+n2}(y) = \frac{b_0 I_{n1+n2}(u)}{b_0 I_{n1+n2}(u)} \quad (33)$$

where $I_{n1+n2}(D_{n1+n2}(y))$ represents the fractional integration of $D_{n1+n2}(y)$ with order $n1+n2$.

This allows us to write:

$$I_n D_n(y) = y(t) \quad (34)$$

Consequently:

$$y + a_1 I_{n2}(y) + a_0 I_{n1+n2}(y) = b_0 I_{n1+n2}(u) \quad (35)$$

The corresponding matrix transformation of equation (35) is given by:

$$y(t) = [-I_{n1+n2}(y) - I_{n2}(y) \ I_{n1+n2}(u)] \begin{bmatrix} a_0 \\ a_1 \\ b_0 \end{bmatrix} = \varphi(t)^T \theta \quad (36)$$

Where

$$\varphi = [-I_{n1+n2}(y) - I_{n2}(y) \ I_{n1+n2}(u)] \quad (37)$$

and

$$\theta = [a_0 \ a_1 \ b_0] \quad (38)$$

Non-integer integrator $I_{n1+n2}(f)$ (($n1+n2$) is the order of integrator I_{n1+n2} of function f determined by equation (39),

$$I_{n1+n2}(f) = I_{n2}(I_{n1}(f))$$

The system initially at rest is a necessary condition for his identification method.

7. SIMULATION EXAMPLE

The fractional system is modeled based on non-integer integrator which is approximated by a dimensional model system. The quadratic criterion relative J is deduced after determining estimated parameter $\theta(t)$ corresponding to each imposed values $n1$

and $n2$ by RLS-forgetting factor. Then minimizing J the $\hat{\theta}$ values containing $a_0 \ a_1 \ b_0$, the estimated parameters are obtained. Measurement data for identification is considered by noiseless signal generated. Simulations are performed with matlab software.

A. Noiseless identification of fractional model ($H_{n1,n2}$)

In this part, results of noiseless identification results of ($H_{n1,n2}$) model are investigated by RLS-forgetting factor. Methods simulated fractional models are performed by following parameters:

$$a_0=3, a_1=1, b_0=4, n_1=0.5, n_2=0.2.$$

These selected parameters must ensure system stability.

The fractional integrator is simulated in a frequency band $[w_b, w_h]$, where $w_b=10^{-3}$ rds/s, and $w_h=103$ rds/s.

The system is excited by a pseudo random binary signal (PRBS) (amplitude equal to 1) represented in fig. 2. The sampling period is set $T_e=10^{-1}$ s.

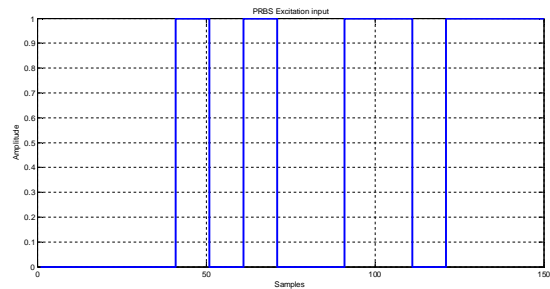


Fig. 2 Pseudo-Random Binary Sequence

Measured output of the system by RLS-forgetting factor method, excited by the same input are shown in figure 3. The insert in figure 3 represents the area of zoom.

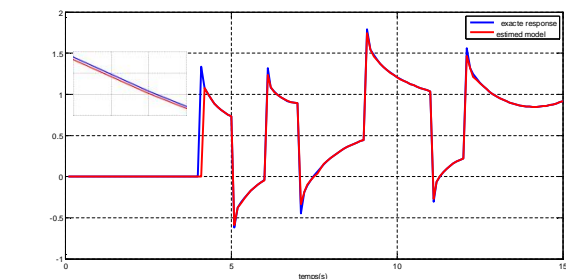


Fig. 3 Measured output and estimated model by RLS-forgetting factor method

The error between the exact response and the estimated models is presented in figure 4.

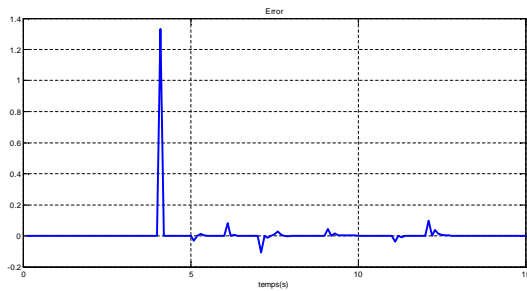


Fig. 4 Error variation between the exact response and the estimated models

The Results of identification of (H_{n_1,n_2}) model are presented in Table 1. It is obvious that the parameters are not converging to the true values of the fractional order model, even if the error between the estimated model and the real one converges to zero. This is a common issue in identification technique and needs further research in order to excite sufficiently the system model [24-25].

Table 1 Noiseless identification of (H_{n_1,n_2}) model estimated by the RLS-forgetting factor method

| Parameters | a_0 | a_1 | b_0 | n_0 | n_1 |
|-----------------|--------|---------|---------|-------|-------|
| Simulation | 3 | 1 | 4 | 0.5 | 0.2 |
| Estimation | 0.0088 | -0.8114 | -2.1487 | 0.5 | 0.2 |
| Quadratic Error | 1.8132 | | | | |

8. CONCLUSION

Fractional order system identification is a hot topic that is focusing a great research effort. In this paper we consider the fractional order systems modeling and identification by recursive least squares (RLS) with forgetting factor technique using the Singularity Function approximation method for fractional order integrators. Identification of fractional differentials equations (FDE) is investigated when estimating system parameters by a linear model with respect to parameters, as well as non-integer orders from temporal data (H_{n_1,n_2}) model. RLS with forgetting factor algorithm was used to estimate parameters. Method and results of simulation are explained in details. It shows that even if the input-output model converges to real plant, there is a parameter bias to be corrected. Further research work will concern the

improvement of the identification model accuracy by developing new estimation techniques.

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