Stabilization of Chaotic Fractional-order Lü System using Sliding Mode Control

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Abstract:
In this paper a fractional-order sliding mode controller (FOSMC) is proposed to control the chaotic fractional-order lü's system. Sliding mode control is a robust controller and its objective is to allo the system states to move to the sliding surface and remain on it. The stability analysis for the proposed control is performed based on the Lyapunov stability theorem. Simulation results show the ability of the proposed controller to improve the system performance.

Keywords: Chaotic System, Fractional Order System, Sliding Mode Control, Lü system.

1. INTRODUCTION

In the recent past there is increasing interest in Fractional Calculus which deals with integration/differentiation of arbitrary orders. Fractional-order dynamic systems, which are as a generalization of integer-order dynamic systems, In provide better mathematical models for some actual physical and engineering systems [1]. We can find many applications in electrochemistry, visco-elasticity, control, and electromagnetic [2,3]. The fractional-order nonlinear dynamic systems have many dynamic behaviors which are similar to the integer-order systems, such as bifurcation, chaos, and attractor [4-5]. The fractional-order chaotic systems are widely studied due to their potential applications in different fields as physics, chemistry, biology, …[2].
Some control methods are proposed to control fractional-order chaotic systems, such as linear feedback controller, backstepping controller, sliding mode method, and the Lyapunov equation-based method [1]. In all these cases, the stability of the whole controlled system has to be analyzed using the fractional-order techniques as well.
The stability of a fractional-order nonlinear time varying system is proposed by Diethelm [6], but the result is valid only for scalar fractional-order systems.

Recently, control and synchronization of fractional–order chaotic systems; using sliding mode control techniques have attracted the attention of many scholars. Sliding mode control is well known for its good robustness against disturbances and parameters uncertainties [7-10].
In this paper a fractional-order sliding mode controller (FOSMC) is proposed to control the chaotic fractional-order lü's system.
Numerical simulations are given for three fractional-order chaotic systems to verify the effectiveness and the universality of the proposed control method.
This paper is organized in the following manner. In Section 2 the preliminaries and some definitions are presented. Some control criteria of fractional-order chaotic systems are proposed in Section 3. In Section 4, numerical simulation of the system shows the effectiveness and the universality of the control method. Finally, conclusions are given in Section 5.

2. PRELIMINARIES ABOUT FRACTIONAL CALCULS

Definitions of Fractional Derivatives and Integrals:

Fractional calculus is a generalization of integration and differentiation to non integer order fundamental operator aD_i^{α}; where a and t are the bounds of the operation and α $\in R$. The continuous integro-differential operator is defined as:

$$aD_i^{\alpha} = \begin{cases} \frac{d^q}{dt^q} & q > 0 \\ 1 & q=1 \\ \int_a^t (d\tau) & q < 0 \end{cases} \quad (1)$$

Two most frequently used definitions for the general fractional differintegral are [6]:

-The Grunwald-Letnikov (GL) definition is formulated as:

$$D^{\alpha} f(t) = \frac{d^{\alpha}}{dt^{\alpha}} f(t) \equiv f^{\alpha}(t) = \lim_{h \to 0} \frac{1}{h^n} \sum_{j=0}^{n} (-1)^j \binom{n}{j} f(t - jh) \; ; \alpha \geq 0$$

$$\binom{n}{j} = \frac{n!}{j!(n-j)!}; \quad (2)$$

-The Riemann-Liouville (RL) is formulated as:

$$D_{t_0}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{t_0}^{t} (t-\tau)^{t-\alpha-1} f(\tau) d\tau \; ; \quad (n-1) < \alpha < n \quad (3)$$

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-1} dt \;\; : \forall x \in \mathbb{R}^* \; ;$$

is gamma Euler function.

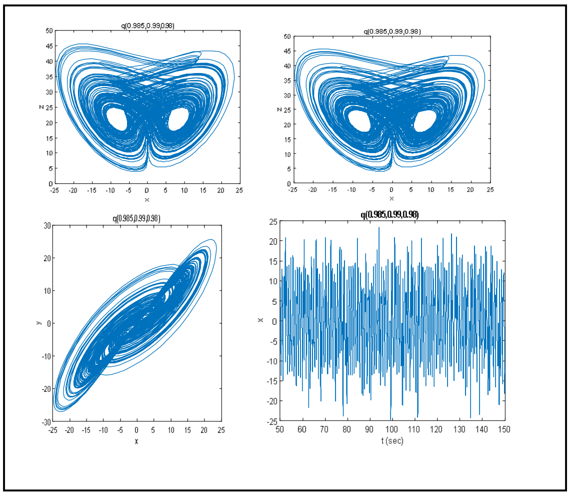

Fig. 1 Lü System chaotic behavior.

Grünwald-Leitnikov (G-L) Approximation

For numerical calculus of fractional-order integrals and derivatives, we can use the G-L definitions and equations.

Thus, for a causal function f(t) and for t = kh, the fractional-order derivative is given as follows [12]:

$$D^{\mu} f(kh) = \frac{d^{\mu}}{dt^{\mu}} f(t) \cong h^{-\mu} \sum_{j=0}^{k} w_j^{(\mu)} f(kh - jh) \quad (4)$$

where,

$$w_j^{(\mu)} = \begin{cases} 1 & j = 0 \\ \left(1 - \frac{1+\mu}{j}\right) & w_j^{(\mu-1)} \;\; j=1,k. \end{cases}$$

3. SYSTEM DESCRIPTION

Consider a class of three dimensional fractional order chaotic system, described as:

$$\begin{cases} D^{(q1)} = -\alpha x + y.f(x,y,z) \\ D^{(q2)} = -\beta y + g(x,y,z) \\ D^{(q3)} = -\gamma z + yh(x,y,z) \end{cases} \quad (5)$$

where q_i(i=1,2,3) are fractional orders satisfing 0< q_i <1 ; x, y, z are state variables ; α, β, ɤ are known constants : U is the contrôler ; each of three fonctions : f(.), g(.), h(.), is considered as continuation nonlinear vector functions belonging to $\mathbb{R}^3 \to \mathbb{R}$ space.

Remark 3.1

The fractional-order system (5) is called a commensurate fractional-order system if, $q_1 = q_2 = q_3 = q$; otherwise, we call the system (5) a non commensurate fractional-order system.

Remark 3.2

Note that many fractional-order chaotic systems belong to the class system characterized by (5) ; Such as: the fractional-order Lorenz, fractional-order Chen ; fractional-order Lu ̈; fractional-order liu system.

It shows that these fractional-order chaotic models can be described by the proposed systems in (5), as summarized in Table 1.

4. CONTROL DESIGN

The control input u(t) is added to the second state equation in order to control chaos in the fractional-order system (5). The proposed class of fractional-order model (5) can be described as:

$$\begin{cases} D^{(q1)} = -\alpha x + y.f(x,y,z) \\ D^{(q2)} = -\beta y + g(x,y,z) + U(t) \\ D^{(q3)} = -\gamma z + yh(x,y,z) \end{cases} \quad (6)$$

Table 1 Class of chaotic systems

Name	model	f (x,y,z)	g (x,y,z)	h (x,y,z)
Chen	$x^{q1} = a(y(t) - x(t))$ $Y^{(q2)} = dx(t) - x(t)z(t) + cy(t)$ $Z^{(q3)} == x(t)y(t) - bz(t)$	a	dx-xz	X
Lorenz	$x^{q1} = a(y(t) - x(t))$ $Y^{(q2)} == x(t)(b - z(t)) - y(t)$ $Z^{(q3)} = x(t)y(t) - cz(t)$	a	x (b-z)	X
Lu	$X^{(q1)} = a(y(t) - x(t))$ $Y^{(q2)} = cy(t) - x(t)z(t)$ $Z^{(q3)} = x(t)y(t) - bz(t)$	a	-xz	X
liu	$X^{(q1)} = -ax(t) - ey^2(t)$ $Y^{(q2)} = by(t) - kx(t)z(t)$ $Z^{(q3)} = -cz(t) + mx(t)y(t)$	-ey	-kxz	Mx

In order to design the sliding-mode controller, the proposed fractional sliding surface is defined as:

$$S(t) = D^{q2-1}(y) + \int_0^t \phi(\tau)\, d\tau \quad (7)$$

where $\phi(t)$ is a function described by:

$$\phi(t) = x.f(x,y,z) + z.h(x,y,z) + \beta.y$$

The sliding mode control will be designed in two phases:

The reaching phase and the sliding phase:

When the fractional-order system operates in sliding mode the sliding surface $S(t)$ and its derivative $\dot{S}(t)$ must satisfied:

$$S(t) = D^{q2-1}(y) + \int_0^t \phi(\tau)\, d\tau = 0$$

$$\dot{S}(t) = D^{q2-1}(y) + \phi(t) = 0 \quad (8)$$

So from the second equation in the system (5) we obtained:

$$D^{q2-1}(t) = -\phi(t) = -x.f(x,y,z) - z.h(x,y,z) - \beta.y \quad (9)$$

$U(t)$ is the total control :

$$U = U_{eq} + U_{sw} \quad (10)$$

U_{eq} is the equivalent control ; U_{sw} is the switching control of the system.

The equivalent control U_{eq} is derived from the system (5) as:

$$U_{eq} = D^{q2-1}(y) - g(x,y,z) - \beta.y \quad (11)$$

The switching control can be chosen as:

$$U_{sw} = k.sgn(S); \quad (12)$$

where:

$$sgn(S)(t) = \begin{cases} -1 & S(t) < 0 \\ 0 & S(t) = 0 \\ 1 & S(t) > 0 \end{cases}$$

Where k is the gain of the controller (k<0), Then;

$$U = D^{q2-1}y(t) - g(x,y,z) - \beta.y + k.sgn(S) \quad (13)$$

5. STABILITY ANALYSIS

Theorem :

The nominal fractional chaotic system in Equation (5) is asymptotically stabilized; under the proposed sliding control law in Equation (13).

Proof :

Define the Lyapunov function candidate:

$$V = \frac{1}{2}S^2 \quad (14)$$

The time derivative of V is given by:

$$\dot{V} = S\dot{S} = S\left(D^{q2-1}(y) + \phi(t)\right)$$

$$\dot{V} = S(g(x,y,z) - \beta y + u(t) + x \cdot f(x,y,z) + zh(x,y,z) + \beta y)$$

$$\dot{V} = S(k.Sgn(S)) = k.|S| < 0. \quad (15)$$

The Lyapunov fonction satisfies the condition of Lyapunov theorem ($V>0$; $\dot{V}<0$) ; the system in the presence of the control is globally asymptotically stable.

6. SIMULATION RESULTS

In this Section we will apply the proposed sliding mode controller to the fractional-order Lü system given by:

$$\begin{aligned} X^{(q1)} &= a(y(t) - x(t)) \\ Y^{(q2)} &= cy(t) - x(t)z(t) \\ Z^{(q3)} &= x(t)y(t) - bz(t) \end{aligned} \quad (16)$$

With : $a = 36$; $b = 3$; $c = 20$;

where x(t), y(t) and z(t) are state variables and $0 < q_1, q_2, q_3 < 1$ are fractional orders.

The simulation results are carried out using the MATLAB software.

For the fractional orders values $q_1 = 0.985$ and $q_2 = 0.95$; $q_3 = 0.98$,

The system in Equations (5) without the controller exhibits a chaotic behavior, as shown in Fig. 1, when (a, b, c,) = (36, 3, 20).

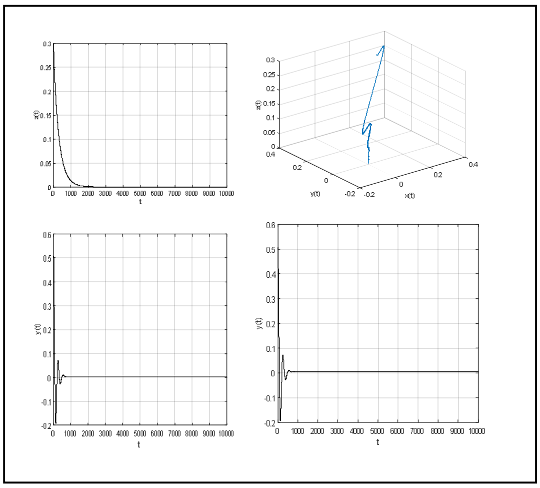

Fig. 2 Controlled system responses.

$$\begin{aligned} X^{(q1)} &= a(y(t) - x(t)) \\ Y^{(q2)} &= cy(t) - x(t)z(t) + U(t) \\ Z^{(q3)} &= x(t)y(t) - bz(t) \end{aligned} \quad (17)$$

With : $a = 36$; $b = 3$; $c = 20$;

To propose a sliding mode control scheme, we define the sliding surface as follows [4,11]:

Sliding surface :

$$S(t) = D^{q-2}\, y(t) + \int_0^t \psi(\tau)d\tau \quad (18)$$

with

$$\psi(t) = xf(x,y,z) + z(t)h(x,y,z) + \beta y(t).$$

$$\psi(t) = ax(t) + xz(t) + cy(t)$$

$$\psi(t) = 36x(t) + xz(t) + 20y(t)$$

The contoller U:

$$U = u_{eq} + u_{sw}$$

$$U(t) = D^{q2}y(t) + x(t)z(t) + \beta y(t) + ksgn(s). \quad k < 0$$

$$U(t) = -ax(t) + ksgn(s) \tag{19}$$

Thus,

$$U(t) = -36x(t) + ksgn(s) \tag{20}$$

The simulation results are shown in Fig. 2. It is obvious that the system is stabilized and all the states converge to a fiwed point.

7. CONCLUSION

In this study, a class of fractional-order chaotic systems has been introduced. According to the Lyapunov stability theorem, a sliding mode control law has been designed to control chaos in the systems. Based on the sliding mode control method, the states of the fractional-order system have been stabilized. Finally, numerical simulations on the control of Lü system are given to verify the effectiveness of the proposed control scheme.

References

[1] S. H. Strogatz. Non lineair dynamics and chaos: With Applications To Physics, Biology, Chemistry, And Engineering (Studies in Nonlinearity) 1st Ed. CRC Press, 2000.

[2] T. Kapitaniak. Chaos forengineers ;Theory, Applications and control; Springer, Berlin, 1998.

[3] B. Bourouba, and S. Ladaci. "Stabilization of class of fractional-orderchaotic system via new sliding mode control." 2017 6th International Conference on Systems and Control (ICSC). IEEE, 2017.

[4] D. Chen, Y. Liu, X. Ma, and R. Zhang, “Control of a class of fractional-orderchaoticsystems via sliding mode,” Nonlinear Dyn. Vol. 67, pp. 893–901, 2012.

[5] X.J., Li, J., Chen, G.R. Chaos in the fractional-order unified system and its synchronisation. J. Franklin Inst. 345, 392–401, 2008.

[6] I. Podlubny, Fractional differential equations, San Diego, USA: Academic press, 1999

[7] Yin, C., Zhong, S. M., & Chen, W. F. Design of sliding mode controller for a class of fractional-order chaotic systems. Communications in Nonlinear Science and Numerical Simulation, 17(1), 356-366, 2012.

[8] M. M. Hadji, S. Ladaci, (2023) Stabilization of incommensurate fractional-order uncertain chaotic systems using terminal sliding mode control scheme. In: GAP 11th International Summit Scientific Research Congress, IKSAD Institute. Gaziantep, Turkiye. December 15-17, 2023.

[9] S. Ladaci, K. Rabah, M. Lashab; “Robust synchronization of fractional-order Arneodo chaotic systems using a fractional sliding mode control strategy”. In: Advanced Synchronization Control and Bifurcation of Chaotic Fractional-Order Systems. A. Boulkroune, S. Ladaci. Eds. USA: IGI Global, 2018.

[10] S. Benrabah, S. Ladaci, “Stabilization of fractional order Modified Van der Pol-Duffing Chaotic Oscillator using a state feedback controller”. In: National Conference on Applied Mathematics and Didactics, NCAMD2021. ENSC, Constantine, Algeria. 26 June, 2021.

[11] K. Rabah, S. Ladaci, and M. Lashab “A novel fractional sliding mode control configuration for synchronizing disturbed fractional order chaotic systems,” Pramana - Journal of Physics, vol. 89, no. 46, 2017.

[12] S. Ladaci and A. Charef, “On Fractional Adaptive Control,” Nonlinear Dynamics, vol. 43, no. 4, pp. 365–378, 2006.