Feedback State Space Stabilization of Fractional-order Chaotic Lorenz-84 Atmosphere Model

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Abstract: This paper explores the stabilization of the three-dimensional fractional-order (FO) chaotic Lorenz-84 atmosphere model through control strategies. Employing the Grünwald-Letnikov approximation for fractional integration, the chaotic system is simulated to understand its intricate dynamics. The main focus lies in utilizing state-space feedback control to adapt input signals based on system states, aiming to steer chaotic dynamics towards stability. Additionally, control gains optimization is conducted through particle swarm optimization (PSO) to guarantee robust stabilization of the chaotic atmosphere model.

Keywords: Lorenz-84 Atmosphere Model; Chaotic system; Fractional-order system; stabilization; Fractional Integrator; Particle Swarm Optimization (PSO)

1. INTRODUCTION

For decades, chaotic systems – those that exhibit unpredictable but deterministic behavior – have captivated scientists [1]. Now, a new layer of complexity is emerging: partial chaotic systems. These systems are described by more complex mathematics that go beyond the realm of integer-order derivatives, and exhibit richer dynamics, full of potential applications across various disciplines [2]. However, harnessing this potential depends on our ability to control these complex systems.

Controlling partial chaotic systems represents an exciting frontier with broad implications across engineering, biology, and secure communications [3]. Methodologies include traditional control techniques, partial-order controllers, adaptive strategies, optimization, and chaos synchronization [4]. Challenges include limited understanding, robust controller design, scalability, computational complexity, and experimental validation [2]. Addressing these issues requires multidisciplinary collaboration to unleash their full potential.

Among the countless complex dynamic behaviors observed in the natural sciences, atmospheric models stand out as an exemplary case study. In 1984, Edward Lorenz made modifications to his system of equations, giving birth to the Lorenz-84 model [5]. By taking advantage of hyper-graphs, we detect chaos situations characterized by incorrect values, leading to deeper exploration [6-7].

Our methodology is based on the Grünwald-Letnikov approximation of partial integral to simulate and understand the complex behavior of the system [8]. Our primary goal Reducing the chaotic tendencies inherent in the model and guiding it towards stability using advanced control methodologies.

Central to our approach is the use of state space feedback control, a powerful technique for manipulating system dynamics by modifying input signals based on system states [9]. Furthermore, we harness the capabilities of particle swarm optimization (PSO) - a nature-inspired transformational algorithm - to determine the optimal control gains, ensuring effective stability of the chaotic Lorenz-84 atmospheric model [10].

This paper is organized as follows: Section II presents some basic definitions of fractional calculus. Section III is dedicated to the description of the fractional order Lorenz-84 system. Then, Section IV presents the proposed control strategy. Simulation results are given in Section V whereas Section VI concludes this paper.

2. FRACTIONAL-ORDER OPERATORS

With a 300-year history, fractional calculus extends differentiation/integration for non-integer orders, finding recent application in control systems through definitions such as Caputo and Grünwald-Letnikov derivatives [6,12].

A. *Grunwald-Leitnikov (G-L) Definition*

For q>0, the definition of the G-L fractional order derivative is

$$D_{GL}^{q} f(t) = \lim_{h \to 0} h^{-q} \sum_{j=0}^{h} (-1)^j \binom{q}{j} f(kh - jh) \quad (1)$$

Where h is a sample period, and coefficients:

$$W_j^q = \binom{q}{j} = \frac{\Gamma(q+1)}{\Gamma(j+1)\Gamma(q-j+1)} \quad (2)$$

With $W_0^q = \binom{q}{0} = 1$are the following binomial coefficients :

$$(1-z)^q = \sum_{j=0}^{\infty} (-1)^j \binom{q}{j} z^j = \sum_{j=0}^{\infty} W_j^{(q)} z^j \quad (3)$$

The fractional G-L integration is formulated as:

$$I_{GL}^{\gamma} f(t) = \lim_{h \to 0} h^{\gamma} \sum_{j=0}^{h} (-1)^j \binom{-\gamma}{j} f(kh - jh) \quad (4)$$

With $W_0^{-\gamma} = \binom{-\gamma}{0} = 1$are the following binomial coefficients :

$$(1-z)^{-\gamma} = \sum_{j=0}^{\infty} (-1)^j \binom{-\gamma}{j} z^j = \sum_{j=0}^{\infty} W_j^{(-\gamma)} z^j \quad (5)$$

B. *Grunwald-Leitnikov (G-L) approximation*

Due to sampling in industrial control, numerical approximations are essential for capturing the fractional dynamics.

The Grünwald-Letnikov (G-L) approximation for the fractional derivative integral of causal function f(t) at time t=kh is given by:

$$D^q f(kh) \simeq h^{-q} \sum_{j=0}^{k} w_j^q f(kh - jh) \quad (6)$$

$$I_{GL}^{\gamma} f(kh) \simeq h^{\gamma} \sum_{j=0}^{k} w_j^{-\gamma} f(kh - jh) \quad (7)$$

The coefficients $w_j^q$and $w_j^{-\gamma}$ represent binomial coefficients from the expressions (8) and (10), respectively. These coefficients can be computed using the following two recursive formulas for j = 1, 2, ..., k.

$$w_0^q = w_0^{-\gamma} = 1$$

$$w_j^q = \left(1 - \frac{1+q}{j}\right) w_{j-1}^q, \; w_j^{-\gamma} = \left(1 - \frac{1-\gamma}{j}\right) w_{j-1}^{-\gamma}$$

3. FRACTIONAL-ORDER CHAOTIC LORENZ-84 MODEL.

The equations below define the fractional Lorenz-84 system [9]:

$$\begin{aligned} D^q x_1(t) &= -a x_1 - x_2^2 - x_3^2 + aF \\ D^q x_2(t) &= -x_2 - x_1 x_2 - b x_1 x_3 + G \\ D^q x_1(t) &= -x_3 + b x_1 x_2 + x_1 x_3 \end{aligned} \quad (8)$$

- x_1, x_2, and x_3 represent global westerly current, cosine phase strength, and sine phase strength, respectively.
- Parameters a and b represent advection of wave strength by easterly and westerly currents, respectively.
- F and G are positive thermal forcing terms that typically depend on temperature constants

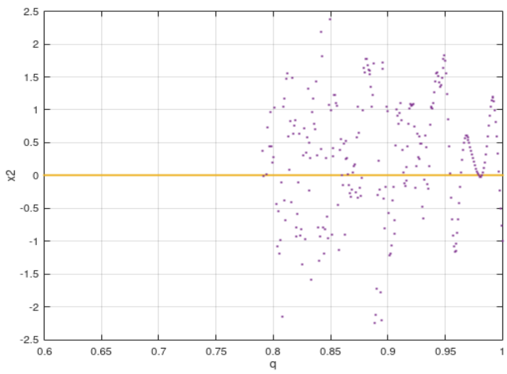

Figure 1. Bifurcation diagram x_2=f(q)

By a simple analysis, we obtain an unstable equilibrium point.

E = (7.99626863, 0.00652641402, 0.029837024).

To select the appropriate fractional order, we analyze the bifurcation diagram that shows the evolution of x_2 with respect to q in Figure 1.

When (F, G, a, b) = (8, 1, 0.25, 4) [13], the diagram allows us to identify the interval over which chaos exists. In our investigation, we selected a specific fractional system (q = 0.83) and analyzed the resulting attractor, which is shown in Figure 2 and Figure 3. The initial conditions used were x(0) = 1, y(0) = 1, and z(0) = 1.

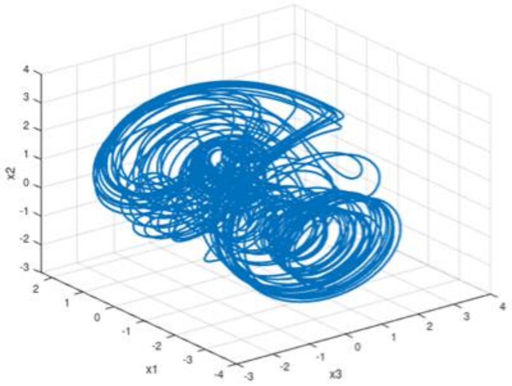

Figure 2. $v(t+1)_{ij} = w.v(t)_{ij} + c_1 r_1 pbest(t)_{ij} - x(t)_{ij} + c_2 r_2 (gbest_j - x(t)_{ij})$Phase plane of fractional chaotic Lorenz-84

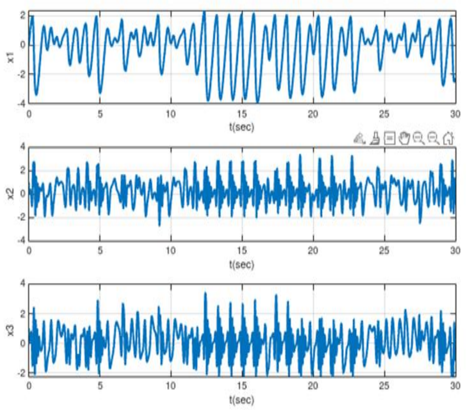

Figure 3. State Variables of Fractional Lorenz-84 System

4. CONTROL STRATEGY

To control chaos in the Lorenz-84 model, we use a combination of State Feedback Control and Particle Swarm Optimization (PSO) based on ITAE as the key function.

A. State Space Feedback Control

Guiding a dynamic system to an equilibrium point using state-space equations and feedback control, where the control input is

$$U = kx \tag{9}$$

K is the feedback gain matrix.

B. Particle Swarm Optimization (PSO)

An algorithm inspired by animal social behavior optimizes control gains for a state-space feedback controller, stabilizing the FO-chaotic Lorenz-84 model. This is achieved through iterative optimization of particle velocities and positions, guided by individual and neighbor experiences as follows [6]:

Velocity update:

$$v(t+1)_{ij} = w.v(t)_{ij} + c_1 r_1 pbest(t)_{ij} - x(t)_{ij} + c_2 r_2 (gbest_j - x(t)_{ij}) \tag{10}$$

Position update:

$$x(t+1)_{ij} = x(t)_{ij} + v(t+1)_{ij} \tag{11}$$

Where:

- $v(t)_{ij}$ is the velocity of particle i in dimension j at time t.
- $x(t)_{ij}$is the position of particle i in dimension j at time t.
- wis the inertia weight, controlling the influence of the previous velocity.
- c_1 and c_2 are acceleration coefficients representing the cognitive and social components, respectively.
- r_1 and r_2 are random numbers sampled from a uniform distribution in the range [0, 1].
- $pbest_{ij}$ is the best position achieved by particle i in dimension j so far.
- $gbest_j$ is the best position achieved by any particle in dimension j so far.

1) Integral Time Absolute Error (ITAE)

The system's performance is evaluated by looking at the total error over time, calculated using the formula [14]:

$$ITAE = \int_0^\infty |e(t)|\, t dt \tag{12}$$

where e(t) is the error at a given time.

5. SIMULATION RESULTS

The following model describes the controlled fractional chaotic Lorenz-84 system incorporating the previously defined controller:

$$\frac{dx_1^q}{dt^q} = -ax_1 - x_2^2 - x_3^2 + aF - k_1 x_1$$

$$\frac{dx_2^q}{dt^q} = -x_2 - x_1 x_2 - bx_1 x_3 + G - k_2 x_2 \tag{13}$$

$$\frac{dx_3^q}{dt^q} = -x_3 + bx_1x_2 + x_1x_3 - k_3x_3$$

In this study, we look for the stabilization of the fractional chaotic Lorenz-84 system on its unstable equilibrium point.

In order to find the best values for controller parameters k_1, k_2, and k_3, we employed a PSO algorithm. The optimized values of k are shown in Figure 4 and the corresponding reduction in error is evident in Figure 5.

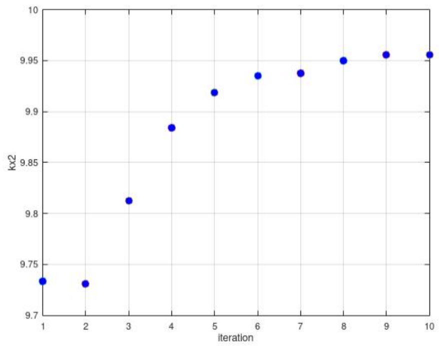

Figure 4. Optimized Controller Parameters Using PSO Algorithm

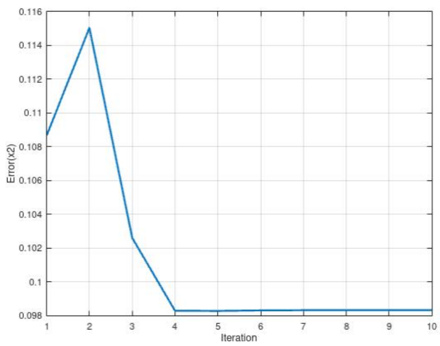

Figure 5. Reduction in Error Corresponding to Optimized Controller Parameters

We utilize feedback with gains k_1= 0.9, k_2 = 0.01, and k_3= 0.05 to stabilize the Lorenz-84 fractional chaotic system at a desired equilibrium point. The results are presented in Figure 6 and Figure 7.

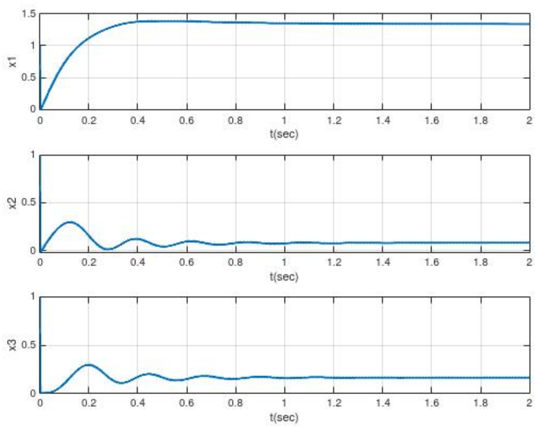

Figure 6. State Variables of Fractional Lorenz-84 System with Control

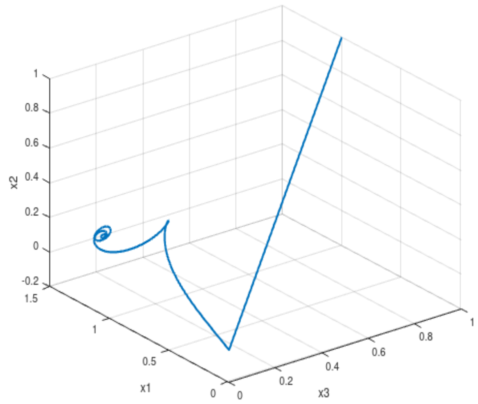

Figure 7. Phase plane of fractional chaotic Lorenz-84 with Control

Simulation results show that the proposed controller is able to stabilize the fractional order chaotic system within a reasonable time.

6. CONCLUSION

This study explores stabilizing the three-dimensional fractional-order chaotic Lorenz-84 atmosphere model using control strategies. We employ the Grünwald-Letnikov approximation for fractional integration to accurately capture system dynamics. State-space feedback control adjusts input signals based on system states, guiding chaotic dynamics towards stability. Additionally, Particle Swarm Optimization (PSO) optimizes control gains to ensure robust stabilization. Results illustrate the effectiveness of state-space feedback control and PSO in stabilizing chaotic behavior. This research significantly contributes to understanding the control of fractional-order chaotic systems.

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